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PRODUCT OF HYPERFUNCTIONS ON THE CIRCLE

BY

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ABSTRACT

Let φ and ψ be two hyperfunctions on the circle which have disjoint support. We interpret in terms of Fourier coefficients the fact that their product, defined in the sense of sheaf theory, vanishes.

1. Introduction

Let $\mathrm{HF}(\mathbb{T})$ be the set of all hyperfunctions on the unit circle $\mathbb{T}$. These objects can be interpreted as linear functionals on the space $\mathcal{O}(\mathbb{T})$ of germs of analytic functions on $\mathbb{T}$ [3, Chap. 1], or as analytic functions on $\mathbb{C} \setminus \mathbb{T}$ vanishing at infinity (see section 2), and are natural generalizations of Schwartz distributions on $\mathbb{T}$. Of course, it is not possible in general to define the product of two hyperfunctions, but this product makes sense for $\varphi \in \mathrm{HF}(\mathbb{T}), \psi \in \mathrm{HF}(\mathbb{T})$ if it is possible to compute in some sense the convolution $\widehat{\varphi}\star\widehat{\psi}$ and if the sequence $((\widehat{\varphi}\star\widehat{\psi})(n))_{n\in\mathbb{Z}}$ is the sequence of Fourier coefficients of some hyperfunction, which will be called the product of φ and ψ. This is the case, for example, if φ is the hyperfunction associated to some function analytic on a neighborhood of the unit circle. We point out in this paper a natural result (corollary 2.7): if φ and ψ are hyperfunctions on $\mathbb{T}$ with disjoint support then

$$\lim_{r\to 1^-} \sum_{n\in\mathbb{Z}} r^{|p|}\widehat{\varphi}(p)\widehat{\psi}(n-p) = 0.$$

If, further, $\lim_{|p|\to+\infty} \widehat{\varphi}(p)\widehat{\psi}(n-p) = 0$, then

$$\lim_{m\to+\infty} \sum_{|p|\le m} \widehat{\varphi}(p)\widehat{\psi}(n-p) = 0.$$

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This result was suggested to the first author by an elegant argument based on the Poisson summation formula and the Paley–Wiener theorem, at the end of a recent paper by Domar [7] where nontrivial invariant subspaces of $\ell^2_\omega(\mathbb{Z})$ are constructed for weights ω such that $\omega(n)\omega(-n) = 1$, $(n \in \mathbb{Z})$ but for which

$$\sum_{n=1}^{\infty} \frac{|\log \omega(n)|}{n^2} = +\infty,$$

provided that $\sum_{n\in\mathbb{Z}} |\log\omega(n+1) + \log\omega(n-1) - 2\log\omega(n)| < +\infty$. Atzmon [2] noticed that Domar's argument shows in fact that if $\phi, \psi \in \mathrm{HF}(\mathbb{T})$, if $\sum_{p\in\mathbb{Z}} |\widehat{\phi}(p)||\widehat{\psi}(n-p)| < +\infty (n \in \mathbb{Z})$ and if $\operatorname{Supp}\phi$ and $\operatorname{Supp}\psi$ lie in disjoint arcs of the unit circle, then $\phi.\psi = 0$, and used this fact and the Beurling–Malliavin theorem [4] to replace the condition $\omega(n)\omega(-n) = 1$ by the condition

$$\sum_{n=1}^{\infty} \frac{|\log \omega(n)\omega(-n)|}{n^2} < +\infty$$

in Domar's theorem (see also [1], [8], [9], [10] and [11] for other recent results about translation invariant subspaces of $\ell^2_\omega(\mathbb{Z})$). Our original proof of Corollary 2.7 was based on standard results about entire functions of exponential type and their Borel and $\mathcal{G}$-transforms [5], but something looked unnatural, and the authors had the feeling that it was possible to see directly that the product $\varphi.\psi$ vanishes in some trivial sense if $\operatorname{Supp}\varphi \cap \operatorname{Supp}\psi = \emptyset$, and then draw conclusions about Fourier coefficients. We found the explanation in the prehistory of microlocal calculus [6]. Hyperfunctions on the circle form a sheaf, and it is possible to define "locally" the product of a hyperfunction and an analytic function. Now let $\varphi, \psi \in \mathrm{HF}(\mathbb{T})$ and assume that the regular points of $\varphi^+, \psi^+, \varphi^-, \psi^-$ satisfy $\operatorname{Reg}(\varphi^+) \cup \operatorname{Reg}(\psi^-) = \mathbb{T}, \operatorname{Reg}(\psi^+) \cup \operatorname{Reg}(\varphi^-) = \mathbb{T}$. Then the product $\varphi.\psi$ can be defined locally everywhere, hence globally since hyperfunctions on the circle form a sheaf (Definition 2.1). Now if φ and ψ are multipliable in this sense, and if we define φ_λ and ψ_λ for $\lambda \in \mathbb{D}^* = \mathbb{D} \setminus \{0\}$ by the formulae $\widehat{\varphi_\lambda}(n) = \lambda^{|n|}\widehat{\varphi}(n)$, $\widehat{\psi_\lambda}(n) = \lambda^{|n|}\widehat{\psi}(n)$ then, if $h \in \mathcal{O}(\mathbb{T})$, the map $(\lambda, \mu) \longmapsto (\varphi_\lambda.\psi_\mu, h)$ has an analytic extension to $(\mathbb{D}^* \cup \Omega) \times (\mathbb{D}^* \cup \Omega)$ where Ω is some domain in $\mathbb{C}$ containing the complex number 1 (see Remark 2.6). It easily follows from this observation that

$$\widehat{\varphi.\psi}(n) = \lim_{r\to 1^-} \sum_{p\in\mathbb{Z}} r^{|p|}\widehat{\varphi}(p)\widehat{\psi}(n-p).$$

Also it follows from the Fatou–Riesz theorem that

$$\widehat{\varphi.\psi}(n) = \lim_{m\to+\infty} \sum_{|p|\le m} \widehat{\varphi}(p)\widehat{\psi}(n-p)$$

if $\lim_{p\to+\infty}\widehat{\varphi}(p)\widehat{\psi}(n-p)+\widehat{\varphi}(-p)\widehat{\psi}(n+p)=0$. The authors believe that this approach gives the heuristics of the phenomenon pointed out by Domar in [7].

Also, our proofs are based on the use of concrete contour integrals (see Lemma 2.3 and formula (2.27)) which gives some hope to obtain results analogous to Corollary 2.9 with weaker notions of support.

The authors wish to thank A. Atzmon for fruitful exchange of information. We also wish to thank N. Nikolski for suggesting that we consider $\lim_{r\to1^-}\sum_{p\in\mathbb{Z}}r^{|p|}\widehat{\varphi}(p)\widehat{\psi}(n-p)$ to give a sense to the convolution product $\widehat{\varphi}\star\widehat{\psi}$, and T. Ransford for suggesting that we use the Fatou–Riesz theorem.

2. Multipliable hyperfunctions on the circle

We will denote by $\mathcal{H}(W)$ the space of holomorphic functions on an open subset W of $\mathbb{C}$. We will say that an open subset W of $\mathbb{C}$ is **admissible** if $W\cap\mathbb{T}\neq\emptyset$. If W is admissible, we set $W^+=W\cap\mathbb{D}$, $W^-=W\cap(\mathbb{C}\setminus\overline{\mathbb{D}})$.

Let L be a nonempty open subset of $\mathbb{T}$, and denote by $\mathcal{U}_L$ the set of all admissible open subsets W of $\mathbb{C}$ such that $W\cap\mathbb{T}=L$. For $W\in\mathcal{U}_L$ denote by $\mathrm{HF}_W(L)$ the quotient space $\mathcal{H}(W\setminus L)/\mathcal{H}(W)$. If $W_1\subset W_2$, the classical excision theorem shows that the map $f+\mathcal{H}(W_2)\longmapsto f_{|W_1\setminus L}+\mathcal{H}(W_1)$ is an isomorphism from $\mathrm{HF}_{W_2}(L)$ onto $\mathrm{HF}_{W_1}(L)$. Hence we can consider that the space $\mathrm{HF}(L):=\mathrm{HF}_W(L)$ does not depend on the choice of $W\in\mathcal{U}_L$.

If $W\in\mathcal{U}_L$, and if $f\in\mathcal{H}(W^+)$, $g\in\mathcal{H}(W^-)$ are given, we will denote by $(f,g)\in\mathrm{HF}(L)$ the coset $F+\mathcal{H}(W)$ where $F_{|W^+}=f$, $F_{|W^-}=g$. If $L_1\subset L$, then $W_1=L_1\cup(W\setminus L)\in\mathcal{U}_{L_1}$. In this case we will define the restriction map $\mathrm{HF}(L)\longmapsto\mathrm{HF}(L_1)$ by the formula

$$(f,g)_{|L_1}=(f_{|W_1^+},g_{|W_1^-})\quad(W\in\mathcal{U}_L,f\in\mathcal{H}(W^+),g\in\mathcal{H}(W^-)).\tag{2.1}$$

Equipped with the restriction maps, the family $\mathrm{HF}_{\mathbb{T}}=\{\mathrm{HF}(L)\}$ forms as well-known a sheaf. Notice also that it immediately follows from the excision theorem that the sheaf $\mathrm{HF}_{\mathbb{T}}$ is a flabby sheaf, which means that $\mathrm{HF}(\mathbb{T})_{|L}=\mathrm{HF}(L)$ for every nonempty open subset of $\mathbb{T}$. Details about these standard facts can be found in [6, Chap. 1].

Set $\mathcal{H}_o(\mathbb{C}\setminus\overline{\mathbb{D}})=\{g\in\mathcal{H}(\mathbb{C}\setminus\overline{\mathbb{D}})\,|\,\lim_{|z|\to+\infty}g(z)=0\}$. Given $\varphi\in\mathrm{HF}(\mathbb{T})$, there exists a unique $\varphi^+\in\mathcal{H}(\mathbb{D})$ and a unique $\varphi^-\in\mathcal{H}_o(\mathbb{C}\setminus\overline{\mathbb{D}})$ such that $\varphi=(\varphi^+,\varphi^-)$. This trivial special case of the excision theorem follows immediately from the fact that holomorphic functions in an annulus admit a Laurent series expansion. We define the Fourier coefficients of $\varphi\in\mathrm{HF}(\mathbb{T})$ by the formulae

$$\varphi^+(\lambda)=\sum_{n=0}^{\infty}\widehat{\varphi}(n)\lambda^n\qquad(|\lambda|<1)\tag{2.2}$$

and

$$\varphi^-(\lambda) = -\sum_{n<0} \widehat{\varphi}(n)\lambda^n \quad (|\lambda| > 1). \tag{2.3}$$

Let L be an open subset of $\mathbb{T}$, and denote by $\mathcal{O}(L)$ the set of real analytic functions on L, i.e. the set $\bigcup_{W\in\mathcal{U}_L} \mathcal{H}(W)_{|L}$. For $h \in \mathcal{O}(L)$, denote by $\mathcal{V}_h$ the set of open subsets $W \in \mathcal{U}_L$ such that h extends holomorphically to W. Let $h \in \mathcal{O}(L)$ and let $W \in \mathcal{V}_h$. Set

$$h.(f,g) = (hf, hg) \quad (f \in \mathcal{H}(W^+),\ g \in \mathcal{H}(W^-)). \tag{2.4}$$

Formula (2.4) defines the product $h.\varphi$ for $h \in \mathcal{O}(L)$, $\varphi \in \mathrm{HF}(L)$.

Now let $h \in \mathcal{O}(\mathbb{T})$, and let $\varphi = (\varphi^+, \varphi^-) \in \mathrm{HF}(\mathbb{T})$. Denote by $\widehat{h}(n)$ the nth Fourier coefficient of h. Then $h \in \mathcal{H}(W_r)$ for some $r \in (0,1)$ where $\mathcal{H}(W_r) = \{z \in \mathbb{C} |\ r < |z| < r^{-1}\}$, and we have

$$h.\varphi = (h\varphi^+_{|W_r^+}, h\varphi^-_{|W_r^-}), \tag{2.5}$$

and so

$$\widehat{h.\varphi}(n) = \sum_{p\in\mathbb{Z}} \widehat{h}(p)\widehat{\varphi}(n-p) \quad (n \in \mathbb{Z}). \tag{2.6}$$

Notice that the series above is absolutely convergent.

Consider again $h \in \mathcal{O}(L)$, $W \in \mathcal{V}_h$. Define $\widetilde{h} \in \mathrm{HF}(L)$ by the formula

$$\widetilde{h} = (h_{|W^+}, 0) = (0, -h_{|W^-}). \tag{2.7}$$

If $h \in \mathcal{O}(\mathbb{T})$, an immediate verification shows that the hyperfunction $\widetilde{h}$ and the function $h \in \mathcal{O}(\mathbb{T}) \subset L^1(\mathbb{T})$ have the same Fourier coefficients.

In the sequel we will often identify a function $h \in \mathcal{O}(L)$ with the hyperfunction $\widetilde{h} \in \mathrm{HF}(L)$. Notice that if $h \in \mathcal{H}(W_r) \subset \mathcal{O}(\mathbb{T})$ for some $r \in (0,1)$, then $\widetilde{h} = (h^+, h^-)$ whereas $h(z) = h^+(z) - h^-(z)$ for $z \in W_r$. Hence the hyperfunction associated to h represents the "jump" between h^+ and h^- on the circle. Similarly, if $W \in \mathcal{U}_L$, $f \in \mathcal{H}(W^+)$, $g \in \mathcal{H}(W^-)$ then the hyperfunction (f,g) can be interpreted heuristically as the "jump" between f and g on L.

With the above notations denote by $\mathrm{Reg}(f)$ the set of elements ξ of L which are regular for f; this means that there exists $r > 0$ such that f extends analytically to $W^+ \cup D(\xi, r)$. The set $\mathrm{Reg}(g)$ is defined in a similar way.

If $\mathrm{Reg}(f) = L$, we can extend f across L and consider f as an element of $\mathcal{O}(L)$. According to [6, Chap 1] we set in this case

$$(f,0).\varphi = f.\varphi \quad (\varphi \in \mathrm{HF}(L)). \tag{2.8}$$

Similarly, if $\mathrm{Reg}(g) = L$ we can extend g across L and consider g as an element of $\mathcal{O}(L)$. In this case set

$$(0,g).\varphi = -g.\varphi \quad (\varphi \in \mathrm{HF}(L)). \tag{2.9}$$

In formulae (2.8) and (2.9), the products $f.\varphi$ and $g.\varphi$ are defined according to formula (2.6). Another way to interpret these formulae consists in using the following rules:

$$(f,0).(u,0) = (fu,0) \tag{2.10}$$

and

$$(0,g).(0,v) = (0,-gv). \tag{2.11}$$

Now if $\mathrm{Reg}(f) = L$, f extends analytically to $W^+ \cup V$, where $V \in \mathcal{U}_L$ and we have

$$(f,0).(0,v) = (0,-f_{|V^-})(0,v) = (0, fv_{|V^-}). \tag{2.12}$$

Also, if $\mathrm{Reg}(g) = L$, g extends analytically to $W^- \cup V$, where $V \in \mathcal{U}_L$, and we have

$$(u,0).(0,g) = (u,0).(-g_{|V^+},0) = (-ug_{|V^+},0). \tag{2.13}$$

According to these formulae, we can define the product $\varphi.\psi$ for $\varphi = (f_1, g_1) \in \mathrm{HF}(L), \psi = (f_2, g_2) \in \mathrm{HF}(L)$ in four situations:

(a) $\mathrm{Reg}(f_1) = \mathrm{Reg}(f_2) = L$,

(b) $\mathrm{Reg}(f_1) = \mathrm{Reg}(g_1) = L$,

(c) $\mathrm{Reg}(f_2) = \mathrm{Reg}(g_2) = L$,

(d) $\mathrm{Reg}(g_1) = \mathrm{Reg}(g_2) = L$.

In all these situations we set

$$\varphi.\psi = (f_1 f_2, 0) + (0, -g_1 g_2) + (f_1, 0).(0, g_2) + (f_2, 0).(0, g_1).$$

Notice that if $\mathrm{Reg}(f) = \mathrm{Reg}(g) = L$, we can consider both f and g as holomorphic functions on some $V \in \mathcal{U}_L$ and we have

$$(-f.g_{|V^+}, 0) = (0, fg_{|V^-}).$$

So the product $\varphi.\psi$ is well defined if $\varphi \in \mathrm{HF}(L)$ and $\psi \in \mathrm{HF}(L)$ satisfy one of the four previous conditions (a), (b), (c), (d). Clearly, we have

$$\varphi.\psi_{|L_1} = \varphi_{|L_1}.\psi_{|L_1} \quad \text{if } L_1 \subset L. \tag{2.14}$$

If L is an open subset of $\mathbb{T}$ we will denote by $\mathcal{V}(L)$ the set of all non-empty subsets of L. Now let $\varphi, \psi \in \mathrm{HF}(\mathbb{T})$ and assume that

$$\mathrm{Reg}(\varphi^+) \cup \mathrm{Reg}(\psi^-) = \mathrm{Reg}(\psi^+) \cup \mathrm{Reg}(\varphi^-) = \mathbb{T}.$$

Set

$$\begin{aligned} \mathcal{V}_{\varphi,\psi} =& \mathcal{V}(\mathrm{Reg}(\varphi^+) \cap \mathrm{Reg}(\psi^+)) \cup \mathcal{V}(\mathrm{Reg}(\varphi^+) \cap \mathrm{Reg}(\varphi^-)) \\ & \cup \mathcal{V}(\mathrm{Reg}(\psi^+) \cap \mathrm{Reg}(\psi^-)) \cup \mathcal{V}(\mathrm{Reg}(\psi^-) \cap \mathrm{Reg}(\varphi^-)). \end{aligned} \tag{2.15}$$

Then $\mathcal{V}_{\varphi,\psi}$ is a covering of $\mathbb{T}$, and for $L \in \mathcal{V}_{\varphi,\psi}$ the product $\varphi_L.\psi_L$ can be defined as above.

Also, it immediately follows from (2.14) that

$$(\varphi_{|L_1}.\psi_{|L_1})_{L_1\cap L_2} = \varphi_{|L_1\cap L_2}.\psi_{|L_1\cap L_2} = (\varphi_{|L_2}.\psi_{|L_2})_{|L_1\cap L_2}$$

if $L_1 \in \mathcal{V}_{\varphi,\psi}$, $L_2 \in \mathcal{V}_{\varphi,\psi}$ and $L_1 \cap L_2 \neq \emptyset$. Since $\mathrm{HF}_{\mathbb{T}}$ is a sheaf, we can introduce the following notion:

Definition 2.1: Let $\varphi, \psi \in \mathrm{HF}(\mathbb{T})$. We say that φ and ψ are multipliable if $\mathrm{Reg}(\varphi^+) \cup \mathrm{Reg}(\psi^-) = \mathrm{Reg}(\psi^+) \cup \mathrm{Reg}(\varphi^-) = \mathbb{T}$. If φ and ψ are multipliable, we define the product $\varphi.\psi \in \mathrm{HF}(\mathbb{T})$ by the condition $\varphi.\psi_{|L} = \varphi_{|L}.\psi_{|L} \quad (L \in \mathcal{V}_{\varphi,\psi})$.

Remark 2.2: (1) Let $h \in \mathcal{O}(\mathbb{T}), \varphi \in \mathrm{HF}(\mathbb{T})$ and let $\widetilde{h} \in \mathrm{HF}(\mathbb{T})$ be the hyperfunction having the same Fourier coefficients as h. We have $\widetilde{h} = (h^+, h^-)$, where h^+ and h^- are defined by (2.2) and (2.3), and so $\mathbb{T} = \mathrm{Reg}(h^+) = \mathrm{Reg}(h^-) \in \mathcal{V}_{\widetilde{h},\varphi}$. Let $r \in (0,1)$ such that $h \in \mathcal{H}(W_r)$. Then h^+ and h^- extend analytically to $\mathbb{D} \cup W_r$ and $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup W_r$, and so $h^+ \in \mathcal{O}(\mathbb{T}), h^- \in \mathcal{O}(\mathbb{T})$. We have $h = h^+ - h^-$. Using (2.10) and (2.11), we obtain

$$\widetilde{h}.\varphi = (h^+, 0).\varphi + (0, h^-).\varphi = h^+.\varphi - h^-.\varphi.$$

Hence we have

$$\widetilde{h}.\varphi = h.\varphi \quad (h \in \mathcal{O}(\mathbb{T}), \varphi \in \mathrm{HF}(\mathbb{T})). \tag{2.16}$$

We thus see that the identification between h and $\widetilde{h}$ is compatible with computation of products.

(2) Let $\varphi \in \mathrm{HF}(\mathbb{T})$, and denote by L_φ the union of all open subsets L of $\mathbb{T}$ such that $\varphi_{|L} = 0$. Set $\operatorname{Supp}\varphi = \mathbb{T} \setminus L_\varphi$. This notion agrees with the usual notion of support if φ is a distribution, i.e. if there exists $p \geq 1$ such that $|\widehat{\varphi}(n)| = O(|n|^p)$ as $|n| \longrightarrow +\infty$. Clearly, $\operatorname{Supp}\varphi$ is the smallest set E such that φ^+ and φ^- extend analytically each other across $\mathbb{T} \setminus E$, and $L_\varphi \subset \operatorname{Reg}(\varphi^+) \cap \operatorname{Reg}(\varphi^-)$.

Let $\varphi, \psi \in \mathrm{HF}(\mathbb{T})$ and assume that $\operatorname{Supp}\varphi \cap \operatorname{Supp}\psi = \emptyset$. Then $L_\varphi \subset \mathcal{V}_{\varphi,\psi}$ and $L_\psi \subset \mathcal{V}_{\varphi,\psi}$, so that φ and ψ are multipliable, since $L_\varphi \cup L_\psi = \mathbb{T}$. We have $\varphi_{|L_\varphi} = 0$, and $\varphi.\psi_{|L_\varphi} = 0$. Similarly $\varphi.\psi_{|L_\psi} = 0$. We thus see that if $\operatorname{Supp}\varphi \cap \operatorname{Supp}\psi = \emptyset$ then φ and ψ are multipliable, and $\varphi.\psi = 0$.

For $h \in \mathcal{O}(\mathbb{T})$ set

$$\rho(h) = \limsup_{|n| \to +\infty} |\widehat{h}(n)|^{1/|n|},$$

so that h extends analytically to $W_{\rho(h)}$. The space $\mathrm{HF}(\mathbb{T})$ can be viewed as the dual space of $\mathcal{O}(\mathbb{T})$, see [3, Chap. 1]. This duality is implemented by the formula

$$\langle h, \varphi \rangle = \sum_{p \in \mathbb{Z}} \widehat{h}(p)\widehat{\varphi}(-p-1) \quad (h \in \mathcal{O}(\mathbb{T}), \varphi \in \mathrm{HF}(\mathbb{T})). \tag{2.17}$$

We also have, for $f \in \mathcal{O}(\mathbb{T}), \varphi \in \mathrm{HF}(\mathbb{T})$, $\rho(h) < r < 1 < R < \rho(h)^{-1}$,

$$\langle h, \varphi \rangle = \frac{1}{2i\pi}\left[\int_{r\mathbb{T}} h(\xi)\varphi^+(\xi)d\xi - \int_{R\mathbb{T}} h(\xi)\varphi^-(\xi)d\xi\right]. \tag{2.18}$$

The difficulty with products of hyperfunctions consists in computing products of the form $(f,0).(0,g)$ where $f \in \mathcal{H}(\mathbb{D})$, $g \in \mathcal{H}_o(\mathbb{C} \setminus \overline{\mathbb{D}})$. The following lemma gives an explicit way to do this:

Lemma 2.3: *Let $f \in \mathcal{H}(\mathbb{D})$, $g \in \mathcal{H}_o(\mathbb{C} \setminus \overline{\mathbb{D}})$ and assume that $\operatorname{Reg}(f) \cup \operatorname{Reg}(g) = \mathbb{T}$. Let $L_f \subset \operatorname{Reg}(f)$ and $L_g \subset \operatorname{Reg}(g)$ be two open sets such that $L_f \cup L_g = \mathbb{T}$, and let $V_f \in \mathcal{U}_{L_f}$ and $V_g \in \mathcal{U}_{L_g}$ be two simply connected open sets such that f extends analytically to $\mathbb{D} \cup V_f$ and g extends analytically to $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup V_g$. Then for every $h \in \mathcal{O}(\mathbb{T})$ we have*

$$\langle h, (f,0).(0,g) \rangle = -\frac{1}{2i\pi}\int_{\Gamma_h} f(\xi)g(\xi)h(\xi)d\xi,$$

where Γ_h is any piecewise-$\mathcal{C}^1$ Jordan curve contained in $\left[V_f^- \cup V_g^+ \cup (L_f \cap L_g)\right] \cap W_{\rho(h)}$ and containing the origin in its interior.

Proof: Set $U = V_f^- \cup V_g^+ \cup (L_f \cap L_g)$. Then U is open and $U \in \mathcal{U}_{L_f \cap L_g}$. Let $\varphi \in \mathrm{HF}(\mathbb{T})$ be defined by the formula

$$\langle h, \varphi\rangle = -\frac{1}{2i\pi}\int_{\Gamma_h} f(\xi)g(\xi)h(\xi)d\xi, \tag{2.19}$$

where Γ_h is any piecewise-$\mathcal{C}^1$ Jordan curve contained in U and containing the origin in its interior.

It follows from Cauchy's theorem that this definition does not depend on the choice of Γ_h. Let $z \in \mathbb{C} \setminus \mathbb{T}$ and set

$$h_z(\xi) = \frac{1}{\xi - z} \quad \text{for } \xi \in \mathbb{T}.$$

Then $h_z \in \mathcal{O}(\mathbb{T})$ and we have

$$\varphi^+(z) = \langle h_z, \varphi\rangle = \frac{1}{2i\pi}\int_{F_z} \frac{f(\xi)g(\xi)}{z-\xi}d\xi \quad (|z| < 1), \tag{2.20}$$

where $F_z \subset U$ is any piecewise-$\mathcal{C}^1$ Jordan curve such that $z \in \operatorname{Int} F_z$.

Similarly we have

$$\varphi^-(z) = \langle h_z, \varphi\rangle = \frac{1}{2i\pi}\int_{G_z} \frac{f(\xi)g(\xi)}{z-\xi}d\xi \quad (|z| > 1), \tag{2.21}$$

where $G_z \subset U$ is any piecewise-$\mathcal{C}^1$ Jordan curve such that z is exterior to G_z. Using formulae (2.20) and (2.21) we extend analytically φ^+ to $\mathbb{D} \cup V_f$ and φ^- to $\mathbb{D} \cup V_g$.

Let $z \in V_f^-$ and let F_z be as above. Let U_z be the component of V_f^- containing z. Then $F_z \cap U_z \neq \emptyset$, for, otherwise, $\mathbb{C} \setminus (\mathbb{D} \cup V_f^-)$ would be contained in $\operatorname{Int} F_z$. Select a closed arc $\Gamma_1 \subset F_z \cap U_z$ and let Γ_2 be a piecewise-$\mathcal{C}^1$ path in U_z such that $\Gamma_1 \cap \Gamma_2$ is a Jordan curve containing z in its interior. Then $(F_z \setminus \Gamma_1) \cup \Gamma_2 = G_z$ is a piecewise-$\mathcal{C}^1$ Jordan curve contained in U satisfying the condition of (2.21). Hence

$$\varphi^+(z) - \varphi^-(z) = \frac{1}{2i\pi}\int_{\Gamma_1 \cup \Gamma_2} \frac{f(\xi)g(\xi)}{z-\xi}d\xi.$$

By Cauchy's formula, we obtain

$$\varphi^+(z) = \varphi^-(z) - f(z)g(z) \quad (z \in V_f^-). \tag{2.22}$$

By a similar argument, we also obtain

$$\varphi^+(z) = \varphi^-(z) - f(z)g(z) \quad (z \in V_g^+). \tag{2.23}$$

Since $\varphi^+ \in \mathcal{H}(V_f \cup \mathbb{D})$, it follows from (2.22) that

$$\varphi_{|L_f} = (\varphi^+, \varphi^+_{|V_f^-} + fg_{|V_f^-}) = (0, fg_{|V_f^-}) = (f,0)_{|L_f}.(0,g)_{|L_f},$$

by (2.12).

Since $\varphi^- \in \mathcal{H}((\mathbb{C} \setminus \overline{\mathbb{D}}) \cup V_g)$, it follows from (2.23) that

$$\varphi_{|L_g} = (\varphi^-_{|V_g^+} - fg_{|V_g^+}, \varphi^-) = (-fg_{|V_g^+}, 0) = (f,0)_{|L_g}.(0,g)_{|L_g},$$

by (2.13). Since $L_f \cup L_g = \mathbb{T}$, this shows that $\varphi = (f,0).(0,g)$. ∎

Remark 2.4: It is possible to give an explicit construction of open sets V_f, V_g, L_f, L_g and of curves Γ_h satisfying the conditions of Lemma 2.3. Of course, if $\mathrm{Reg}(f) = \emptyset$ we can take $V_f = \emptyset, V_g = W_\rho$, where $\rho = \rho(h)$ and $\Gamma_h = r\mathbb{T}$, where $\rho < r < 1$. Also, if $\mathrm{Reg}(g) = \emptyset$ we can take $V_f = W_\rho$, where $\rho = \rho(h), V_g = \emptyset$, and $\Gamma_h = R\mathbb{T}$ where $1 < R < \rho^{-1}$.

Now assume that $\mathrm{Reg}(f) \neq \emptyset, \mathrm{Reg}(g) \neq \emptyset$, and that $\mathrm{Reg}(f) \cup \mathrm{Reg}(g) = \mathbb{T}$. Since $\mathbb{T} \setminus \mathrm{Reg}(g)$ is a compact subset of $\mathrm{Reg}(f)$ there exists a finite family $F_1, F_2, \dots, F_k$ of open arcs, with $\overline{F_j} \subset \mathrm{Reg}(f)$ for $j \leq k$, such that $\mathbb{T} \setminus \mathrm{Reg}(g) \subset \bigcup_{j \leq k} F_j$. Taking $F_{j_1} \cup F_{j_2}$ instead of F_{j_1} and F_{j_2} whenever $F_{j_1} \cap F_{j_2} \neq \emptyset$ we can arrange that $F_1, \dots, F_k$ be pairwise disjoint, and taking smaller arcs whenever $\overline{F_{j_1}} \cap \overline{F_{j_2}} \neq \emptyset$ we can in fact arrange that $\overline{F_1}, \cdots, \overline{F_k}$ be pairwise disjoint. Of course, we can assume that none of the arcs $F_1, \dots, F_k$ is contained in $\mathrm{Reg}(g)$.

Denote by $a_j = e^{is_j}$ and by $b_j = e^{it_j}$ the endpoints of F_j. By using a suitable renumbering of the family $(F_j)_{j \leq k}$ we can arrange that $s_1 < t_1 \cdots < s_k < t_k < s_1 + 2\pi$. Set $G_j = \{e^{is}\}_{t_j < s < s_{j+1}}$ for $j \leq k-1, G_k = \{e^{is}\}_{t_k < s < s_1}$. For $j \leq k$ choose an open arc H_j containing a_j and an open arc K_j containing b_j such that $\overline{H_j} \cup \overline{K_j} \subset \mathrm{Reg}(f) \cap \mathrm{Reg}(g)$, so that $\overline{H_1}, \dots, \overline{H_k}, \overline{K_1}, \dots, \overline{K_k}$ are pairwise disjoint. Set

$$L_f = \bigcup_{j \leq k} (F_j \cup H_j \cup K_j), \quad L_g = \bigcup_{j \leq k} (G_j \cup H_j \cup K_j).$$

Then $L_f \cup L_g = \mathbb{T}, \overline{L_f} \subset \mathrm{Reg}(f), \overline{L_g} \subset \mathrm{Reg}(g)$.

If L is an open subset of $\mathbb{T}$, set $V_{L,r} = \{z \in \mathbb{C} | r < |z| < r^{-1}, z/|z| \in L\}$ for $r \in (0,1)$. Since $\overline{L_f} \subset \mathrm{Reg}(f)$, a standard compactness argument shows that there exists $r_1 \in (0,1)$ such that f extends analytically to $\mathbb{D} \cup V_{L_f, r_1}$. Similarly, we see that there exists $r_2 \in (0,1)$ such that g extends analytically to $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup V_{L_g,r_2}$. Set $\delta = \sup(r_1, r_2)$ and, for $r \in (\delta, 1)$, denote by Γ_r the Jordan curve $\bigcup_{j\leq k}(r\overline{G_j} \cup r^{-1}\overline{F_j} \cup [ra_j, r^{-1}a_j] \cup [rb_j, r^{-1}b_j])$. Now let $h \in \mathcal{O}(\mathbb{T})$ and let r be such that $\sup(\rho(h), \delta) < r < 1$. Then the Jordan curve Γ_r satisfies the conditions of Lemma 2.3 with respect to h.

Notice that if $\operatorname{Supp}\varphi \cap \operatorname{Supp}\psi = \emptyset$ we can use the same contours Γ_r to compute $\langle h, (\varphi^+, 0).(0, \psi^-)\rangle$ and $\langle h, (\psi^+, 0).(0, \varphi^-)\rangle$ for $h \in \mathcal{O}(\mathbb{T})$.

THEOREM 2.5: *Let $\varphi, \psi \in \mathrm{HF}(\mathbb{T})$. If φ and ψ are multipliable, then*

$$\widehat{\varphi.\psi}(n) = \lim_{r\to 1^-} \sum_{p\in\mathbb{Z}} r^{|p|}\widehat{\varphi}(p)\widehat{\psi}(n-p) \quad (n \in \mathbb{Z}).$$

If, further, $\lim_{p\to+\infty} \widehat{\varphi}(p)\widehat{\psi}(n-p) + \widehat{\varphi}(-p)\widehat{\psi}(n+p) = 0$, then

$$\widehat{\varphi.\psi}(n) = \lim_{m\to+\infty} \sum_{|p|\leq m} \widehat{\varphi}(p)\widehat{\psi}(n-p).$$

Proof: Set $e_n(\xi) = \xi^{-n-1}$ for $n \in \mathbb{Z}$. It follows from (2.17) that we have

$$\langle e_n, \varphi\rangle = \widehat{\varphi}(n) \quad (n \in \mathbb{Z}, \varphi \in \mathrm{HF}(\mathbb{T})). \tag{2.24}$$

Now for $\varphi \in \mathrm{HF}(\mathbb{T}), \lambda \in \overline{\mathbb{D}}^* := \overline{\mathbb{D}} \setminus \{0\}$, define $\varphi_\lambda \in \mathrm{HF}(\mathbb{T})$ by the formula

$$\widehat{\varphi_\lambda}(n) = \lambda^{|n|}\widehat{\varphi}(n), \tag{2.25}$$

so that $\varphi_\lambda \in \mathcal{O}(\mathbb{T})$ for $\lambda \in \mathbb{D}^*$. We have, for $\varphi \in \mathrm{HF}(\mathbb{T})$, $\psi \in \mathrm{HF}(\mathbb{T})$,

$$\widehat{\varphi_\lambda.\psi_\mu}(n) = \sum_{p\in\mathbb{Z}} \lambda^{|p|}\mu^{|n-p|}\widehat{\varphi}(p)\widehat{\psi}(n-p) \quad ((\lambda,\mu)\in \overline{\mathbb{D}}^* \times \mathbb{D}^* \cup \mathbb{D}^* \times \overline{\mathbb{D}}^*), \tag{2.26}$$

and so the map $(\lambda, \mu) \longmapsto \widehat{\varphi_\lambda.\psi_\mu}(n)$ is analytic on $\mathbb{D}^* \times \mathbb{D}^*$ for every $n \in \mathbb{Z}$.

Now let $\varphi, \psi \in \mathrm{HF}(\mathbb{T})$ and assume that φ and ψ are multipliable. If L is an open subset of $\mathbb{T}$ set again $V_{L,r} = \{z \in \mathbb{C} | r < |z| < r^{-1}, z/|z| \in L\}$ for $r \in (0,1)$. Also, for $\epsilon > 0$ set

$$L^\epsilon = \{\xi \in \mathbb{T} | \inf_{z\in L} |\operatorname{Arg} z - \operatorname{Arg}\xi| < \epsilon\}.$$

It follows from Remark 2.4 that there exists two positive numbers ϵ and δ and four open subsets $L_{\varphi^+}, L_{\psi^-}, L_{\psi^+}, L_{\varphi^-}$ of $\mathbb{T}$ which possess the following properties:

(i) $L_{\varphi^+} \cap L_{\psi^-} = L_{\psi^+} \cap L_{\varphi^-} = \emptyset$, $\quad \overline{L_{\varphi^+}} \cup \overline{L_{\psi^-}} = \overline{L_{\psi^+}} \cup \overline{L_{\varphi^-}} = \mathbb{T}$.

(ii) φ^+ extends analytically to $\mathbb{D} \cup V_{L^\epsilon_{\varphi^+},\delta}$ and ψ^- extends analytically to $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup V_{L^\epsilon_{\psi^-},\delta}$.

(iii) ψ^+ extends analytically to $\mathbb{D} \cup V_{L^\epsilon_{\psi^+},\delta}$ and φ^- extends analytically to $(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup V_{L^\epsilon_{\varphi^-},\delta}$.

Set $r = \sqrt{\delta}$ and for $\delta \le \rho < r$, $0 < \eta \le \epsilon$, $\theta = \varphi^+, \varphi^-, \psi^+, \psi^-$ set $V_{\theta,\eta,\rho} = V_{L^\eta_{\theta,\rho}}$. Set

$$\Gamma_{1,r} = r^{-1}\overline{L_{\varphi^+}} \cup r\overline{L_{\psi^-}} \cup \Big(\bigcup_{r \le s \le r^{-1}} s[\overline{L_{\varphi^+}} \cap \overline{L_{\psi^-}}]\Big)$$

and

$$\Gamma_{2,r} = r^{-1}\overline{L_{\psi^+}} \cup r\overline{L_{\varphi^-}} \cup \Big(\bigcup_{r \le s \le r^{-1}} s[\overline{L_{\psi^+}} \cap \overline{L_{\varphi^-}}]\Big).$$

Also set

$$U_{1,\eta,\rho} = V^+_{\varphi^+,\eta,\rho} \cup V^-_{\psi^-,\eta,\rho} \cup (L^\eta_{\varphi^+} \cap L^\eta_{\psi^-})$$

and

$$U_{2,\eta,\rho} = V^+_{\psi^+,\eta,\rho} \cup V^-_{\varphi^-,\eta,\rho} \cup (L^\eta_{\psi^+} \cap L^\eta_{\varphi^-}).$$

Then $\Gamma_{1,r}$ and $\Gamma_{2,r}$ are Jordan curves respectively contained in $U_{1,\eta,\rho}$ and $U_{2,\eta,\rho}$ for $\delta \le \rho < r, 0 < \eta \le \epsilon$.

Now set $\Omega = \{z \in \mathbb{C} | 0 < |z| < r^{-1}, |\operatorname{Arg} z| < \epsilon\}$. Fix $n \in \mathbb{Z}$. For $\lambda \in \Omega, \mu \in \Omega$ set

$$\begin{aligned} F_n(\lambda,\mu) =& \frac{1}{2i\pi}\int_{r\mathbb{T}} \varphi^+(\lambda\xi).\psi^+(\mu\xi)e_n(\xi)d\xi \\ &+ \frac{1}{2i\pi}\int_{r^{-1}\mathbb{T}} \varphi^-(\lambda^{-1}\xi).\psi^-(\mu^{-1}\xi)e_n(\xi)d\xi \\ &- \frac{1}{2i\pi}\int_{\Gamma_{1,r}} \varphi^+(\lambda\xi).\psi^-(\mu^{-1}\xi)e_n(\xi)d\xi \\ &- \frac{1}{2i\pi}\int_{\Gamma_{2,r}} \psi^+(\mu\xi).\varphi^-(\lambda^{-1}\xi)e_n(\xi)d\xi. \end{aligned} \tag{2.27}$$

For $\lambda \in \mathbb{C}\setminus\{0\}$, set $f_\lambda(\xi) = f(\lambda\xi)$ for $f \in \mathcal{H}(\mathbb{D})$, $g_\lambda(\xi) = g(\lambda^{-1}\xi)$ for $g \in \mathcal{H}_o(\mathbb{C}\setminus\overline{\mathbb{D}})$, so that f_λ is analytic for $|\xi| < |\lambda|^{-1}$ and g_λ is analytic for $|\xi| > |\lambda|$.

Let $\lambda \in \Omega$ and set $\rho = |\lambda|^{-1}, \eta = \epsilon - |\operatorname{Arg}\lambda|$. Then φ^+_λ extends analytically to $\rho[\mathbb{D}^* \cup V^+_{\varphi^+,\eta,\delta}]$, ψ^+_λ extends analytically to $\rho[\mathbb{D}^* \cup V^+_{\psi^+,\eta,\delta}]$, φ^-_λ extends analytically to $\rho^{-1}[(\mathbb{C}\setminus\overline{\mathbb{D}}) \cup V^-_{\varphi^-,\eta,\delta}]$, and ψ^-_λ extends analytically to $\rho^{-1}[(\mathbb{C}\setminus\overline{\mathbb{D}}) \cup V^-_{\psi^-,\eta,\delta}]$.

If $\rho > 1$, then $\rho[\mathbb{D} \cup V^+_{\varphi^+,\eta,\delta}] \cap \rho^{-1}[(\mathbb{C}\setminus\overline{\mathbb{D}}) \cup V^-_{\psi^-,\eta,\delta}]$ contains $U_{1,\eta,\rho^{-1}.\delta}$, and $\rho[\mathbb{D} \cup V^+_{\psi^+,\eta,\delta}] \cap \rho^{-1}[(\mathbb{C}\setminus\overline{\mathbb{D}}) \cup V^-_{\varphi^-,\eta,\delta}]$ contains $U_{2,\eta,\rho^{-1}.\delta}$, and $\Gamma_{j,r} \subset U_{j,\eta,\rho^{-1}.\delta}$, $j = 1,2$, since $\rho^{-1}.\delta < r$. Also $r\mathbb{T} \subset \rho\mathbb{D}^*$ and $r^{-1}\mathbb{T} \subset \rho^{-1}(\mathbb{C}\setminus\overline{\mathbb{D}})$.

If $\rho \le 1$, then $\rho[\mathbb{D}^* \cup V^+_{\varphi^+,\eta,\delta}] \subset \mathbb{D}^* \cup V^+_{\varphi^+,\eta,\delta}$, $\rho[\mathbb{D}^* \cup V^+_{\psi^+,\eta,\delta}] \subset \mathbb{D}^* \cup V^+_{\psi^+,\eta,\delta}$, $\rho^{-1}[(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup V^-_{\varphi^-,\eta,\delta}] \subset (\mathbb{C} \setminus \overline{\mathbb{D}}) \cup V^-_{\varphi^-,\eta,\delta}$, $\rho^{-1}[(\mathbb{C} \setminus \overline{\mathbb{D}}) \cup V^-_{\psi^-,\eta,\delta}] \subset (\mathbb{C} \setminus \overline{\mathbb{D}}) \cup V^-_{\psi^-,\eta,\delta}$. This shows that F_n is well-defined and analytic on $\Omega \times \Omega$.

Since $L_{\varphi^+,\eta} \cup L_{\psi^-,\eta} = L_{\psi^+,\eta} \cup L_{\varphi^-,\eta} = \mathbb{T}$ for $0 < \eta \le \epsilon$, it follows from Lemma 2.3 that $F_n(\lambda, \mu) = \widehat{\varphi_\lambda . \psi_\mu}(n)$ for $\lambda \in \Omega \cap \overline{\mathbb{D}}$, $\mu \in \Omega \cap \overline{\mathbb{D}}$. In particular we see that $\sum_{p \in \mathbb{Z}} \lambda^{|p|} . \mu^{|n-p|} . \widehat{\varphi}(p) . \widehat{\psi}(n-p) = \widehat{\varphi_\lambda . \psi_\mu}(n)$ converges to $\widehat{\varphi . \psi}(n)$ as (λ, μ) converges to $(1,1)$ in $(\overline{\mathbb{D}}^* \times \mathbb{D}^*) \cup (\mathbb{D}^* \times \overline{\mathbb{D}}^*)$.

Now set $b_o = \widehat{\varphi}(0) . \widehat{\psi}(n)$, $b_p = \widehat{\varphi}(p) . \widehat{\psi}(n-p) + \widehat{\varphi}(-p) . \widehat{\psi}(n+p)$ for $p \ge 1$, so that

$$\sum_{p=0}^{+\infty} b_p \lambda^p = \widehat{\varphi_\lambda . \psi}(n) = F_n(\lambda, 1) \quad \text{for } |\lambda| < 1.$$

Since F_n is analytic on $\Omega \times \Omega$, we see that

$$\lim_{r \to 1^-} \sum_{p=0}^{+\infty} b_p r^p e^{ipt} = F_n(e^{it}, 1),$$

uniformly for $|t| \le \epsilon/2$. Recall the following Fatou–Riesz theorem [12, Vol. 1, Chap. 17, p. 404]: If $e^{it} \in \operatorname{Reg} \Theta$, where $\Theta(\lambda) = \sum_{p=0}^{+\infty} a_p \lambda^p$ for $|\lambda| < 1$, and if $\lim_{p \to +\infty} a_p = 0$, then $\sum_{p=0}^{+\infty} a_p e^{ipt}$ is convergent. But e^{it} is a regular point for the series $\sum_{p=0}^{+\infty} b_p \lambda^p$ for $|t| \le \epsilon/2$, and if

$$\lim_{p \to +\infty} b_p = \lim_{p \to +\infty} \widehat{\varphi}(p) . \widehat{\psi}(n-p) + \widehat{\varphi}(-p) . \widehat{\psi}(n+p) = 0,$$

this shows that the series $\sum_{p=0}^{+\infty} b_p$ is convergent, and we have

$$\widehat{\varphi . \psi}(n) = F_n(1,1) = \sum_{p=0}^{+\infty} b_p = \lim_{m \to +\infty} \sum_{|p| \le m} \widehat{\varphi}(p) \widehat{\psi}(n-p). \qquad \blacksquare$$

Remark 2.6: Using the same method it is easy to show that if $h \in \mathcal{O}(\mathbb{T})$ then the map $(\lambda, \mu) \longmapsto \langle \varphi_\lambda . \psi_\mu, h \rangle$ extends analytically to $(\mathbb{D}^* \cup \rho\Omega) \times (\mathbb{D}^* \cup \rho\Omega)$ for some $\rho > 1$ depending on h. Also, if we equip HF($\mathbb{T}$) with the Frechet–Schwartz topology of uniform convergence on bounded subsets of $\mathcal{O}(\mathbb{T})$ then the map $(\lambda, \mu) \longmapsto \varphi_\lambda . \psi \mu$ has a continuous extension to $[(\mathbb{D}^* \cup (\Omega \cap \mathbb{T})] \times [\mathbb{D}^* \cup (\Omega \cap \mathbb{T})]$. Using Remark 2.2 and Theorem 2.5, we obtain

COROLLARY 2.7: *Let $\varphi, \psi \in$ HF($\mathbb{T}$). If $\operatorname{Supp} \varphi \cap \operatorname{Supp} \psi = \emptyset$, then*

$$\lim_{r \to 1^-} \sum_{p \in \mathbb{Z}} r^{|p|} \widehat{\varphi}(p) . \widehat{\psi}(n-p) = 0 \quad (n \in \mathbb{Z}).$$

If, further, $\lim_{p \to +\infty} \widehat{\varphi}(p) . \widehat{\psi}(n-p) + \widehat{\varphi}(-p) . \widehat{\psi}(n+p) = 0$, then

$$\lim_{m \to +\infty} \sum_{|p| \le m} \widehat{\varphi}(p) . \widehat{\psi}(n-p) = 0.$$

References

[1] A. Atzmon, *Entire functions, invariant subspaces and Fourier transforms,* Israel Mathematical Conference Proceedings **11** (1997), 37–52.

[2] A. Atzmon, *Weighted L^p spaces of entire functions, Fourier transforms and invariant subspaces,* preprint.

[3] C. A. Berenstein and R. Gay, *Complex Analysis and Special Topics in Harmonic Analysis,* Vol. 2, Springer-Verlag, Berlin–Heidelberg–New York, 1995.

[4] A. Beurling and P. Malliavin, *On Fourier transforms of measures with compact support,* Acta Mathematica **107** (1962), 291–309.

[5] R. P. Boas, *Entire Functions,* Academic Press, New York, 1952.

[6] A. Cerezo, *Le cas d'une variable complexe,* in *Hyperfunctions and Theoretical Physics,* Lecture Notes in Mathematics **449**, Springer-Verlag, Berlin, 1975, pp. 5–20.

[7] Y. Domar, *Entire functions of order $\leq$ 1, with bounds on both axes,* Annales Academiae Scientiarum Fennicae **22** (1997), 339–348.

[8] J. Esterle, *Singular inner functions and biinvariant subspaces for disymetric weighted shifts,* Journal of Functional Analysis **144** (1997), 64–104.

[9] J. Esterle and A. Volberg, *Sous-espaces invariants par translation de certains espaces de Hilbert de suites quasi-analytiquement pondérées,* Comptes Rendus de l'Académie des Sciences, Paris **326,** Série 1 (1998), 295–300.

[10] J. Esterle and A. Volberg, *Analytic left-invariant subspaces of weighted Hilbert spaces of sequences,* Journal of Operator Theory, to appear.

[11] J. Esterle and A. Volberg, *Asymptotically holomorphic functions and translation invariant subspaces of weighted Hilbert spaces of sequences,* preprint.

[12] A. I. Markushevich, *Theory of Functions of a Complex Variable* (three volumes in one), Chelsea Publishing Company, New York, 1985.